Curl: For $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ - vector field,

$$\operatorname{curl} \vec{F} = \text{"}\nabla \times \vec{F}\text{"} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

mnemonic

If $\vec{F}$ is conservative, then $\operatorname{curl} \vec{F} = 0$.

Moreover, for $\vec{F}$ on entire $\mathbb{R}^3$, if $\operatorname{curl} \vec{F} = 0$ then $\vec{F}$ is conservative.

⚠ $\operatorname{curl}(\nabla f) = \vec{0}$ for any function f.

Divergence:

$$\operatorname{div} \vec{F} = \text{"}\nabla \cdot \vec{F}\text{"} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$\vec{F} = \langle P, Q, R \rangle$

mnemonic

⚠ $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$ for any vector-field $\vec{F}$.

For $\vec{F} = \langle P(x,y), Q(x,y), 0 \rangle$ one has

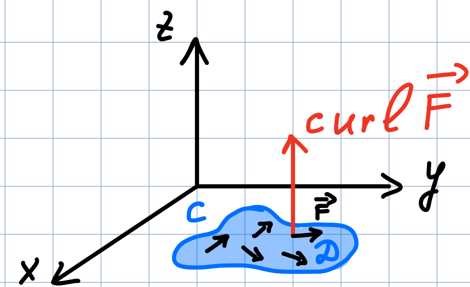
$$\operatorname{curl} \vec{F} = \vec{k} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$\Rightarrow \vec{k} \cdot \operatorname{curl} \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$\boxed{\Rightarrow}$

Green's theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{\mathcal{D}} \underbrace{(\operatorname{curl} \vec{F}) \cdot \vec{k}}_{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}} \, dA$$



$$\operatorname{div} \nabla f = \text{"}\nabla \cdot \nabla f\text{"} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

— Laplace operator.

True or false?

1) $\operatorname{div}(\vec{F}) = 0 \iff \vec{F}$ is conservative

2) $\operatorname{div}(f) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$

3) $\operatorname{div}(\nabla f) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$

$\vec{F} = \langle P, Q, R \rangle$ - vector field

$f(x, y, z)$ - function

$\operatorname{div}(\vec{F}) = 0 \iff \vec{F}$ is conservative **F**

$\operatorname{curl}(\nabla f) = 0 \quad \forall$ function f

$\operatorname{div}(f) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$ **F**

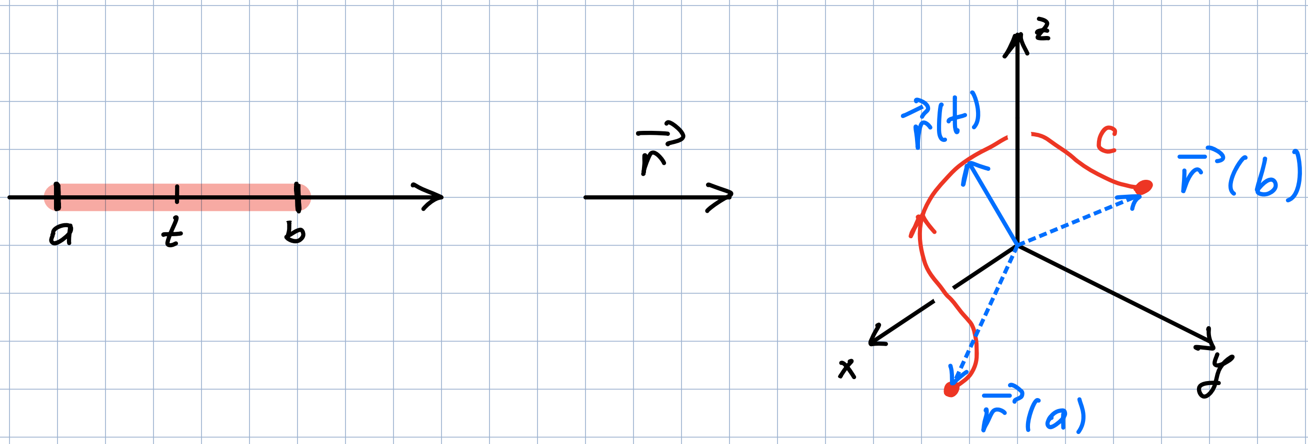
$\operatorname{div} \vec{F} = {}$ " $\vec{\nabla} \cdot \vec{F}$ " $= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$\operatorname{div}(\nabla f) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$ **F**

$\operatorname{div} \nabla f = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

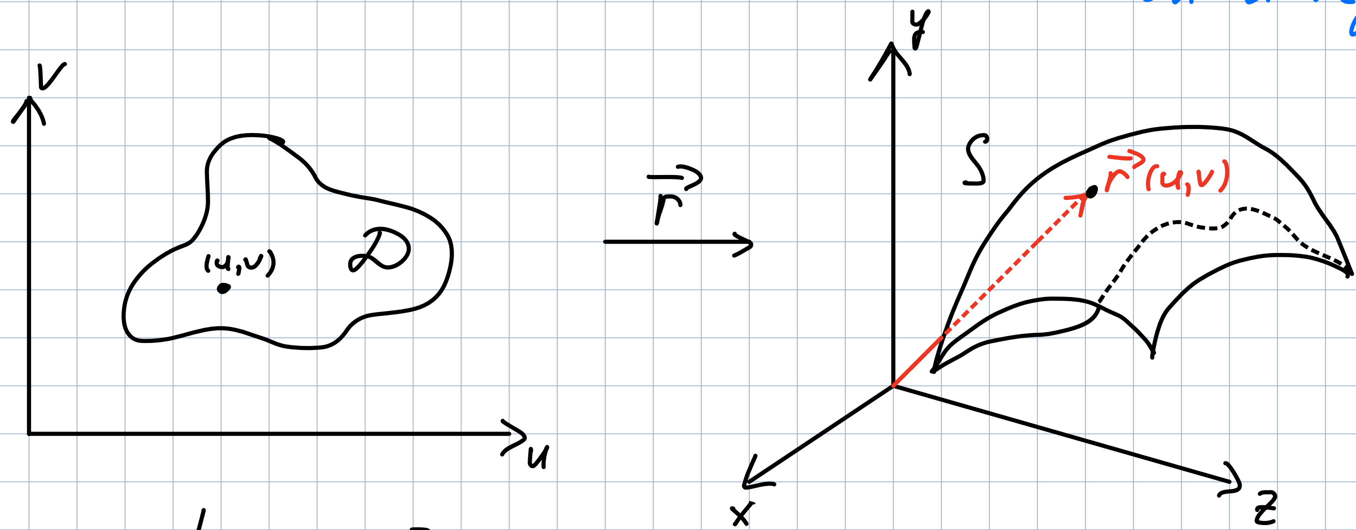
Parametric surfaces

Recall: a curve can be described by $\vec{r}(t)$



- A surface can be described by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

vector-valued fct. of u, v on a region $\mathcal{D}$ in uv-plane



$$S: \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

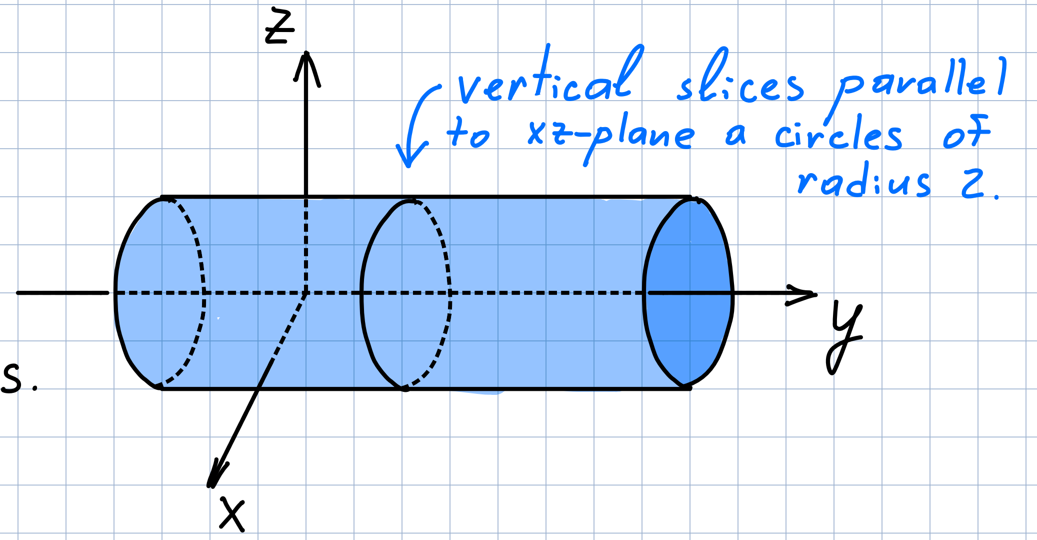
parametric equations

S - parametric surface

Ex: $\vec{r}(u,v) = \langle 2\cos u, v, 2\sin u \rangle$. Sketch the parametric surface.

Sol: $\left.\begin{array}{l} x = 2\cos u \\ y = v \\ z = 2\sin u \end{array}\right\} \Rightarrow \begin{cases} x^2 + z^2 = 4 \\ y = v \end{cases}$

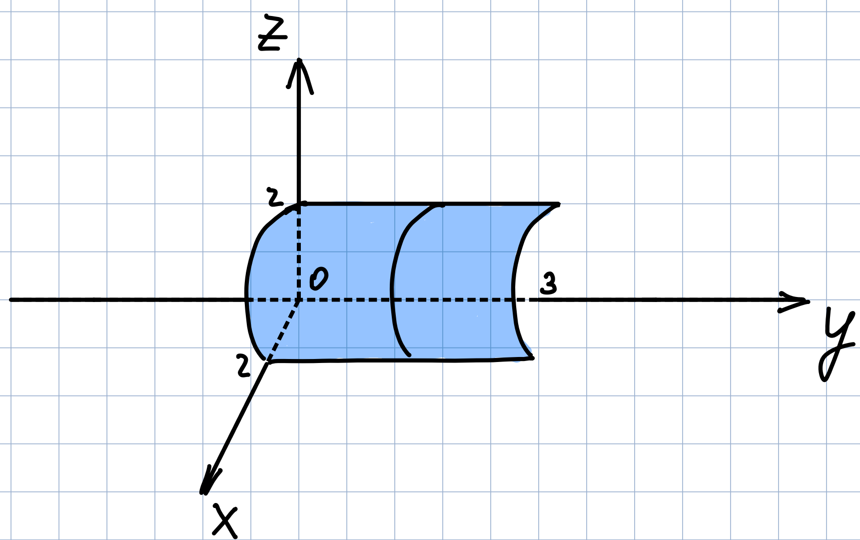
- cylinder of radius 2 along y-axis.



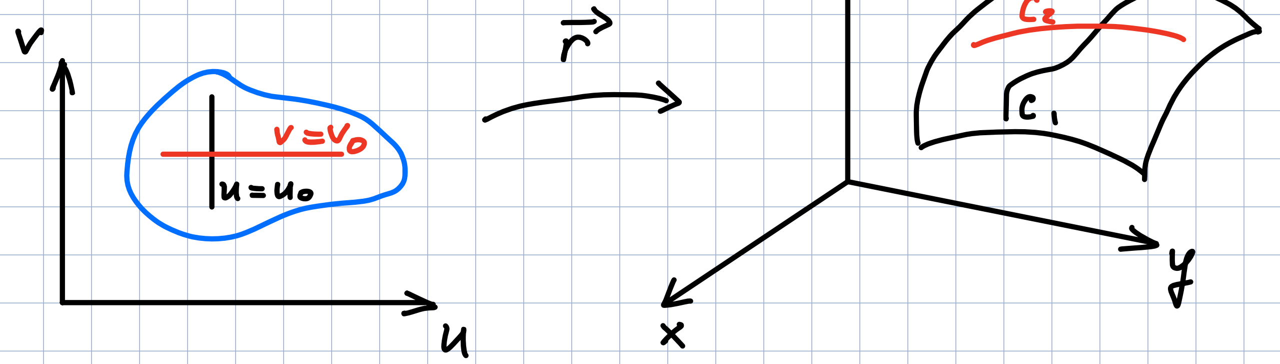
Ex: Imposing restrictions:

$$\begin{cases} 0 \le u \le \frac{\pi}{2} \\ 0 \le v \le 3 \end{cases} \Rightarrow \begin{array}{l} 0 \le y \le 3 \\ x, z \ge 0 \end{array}$$

- quater-cylinder.



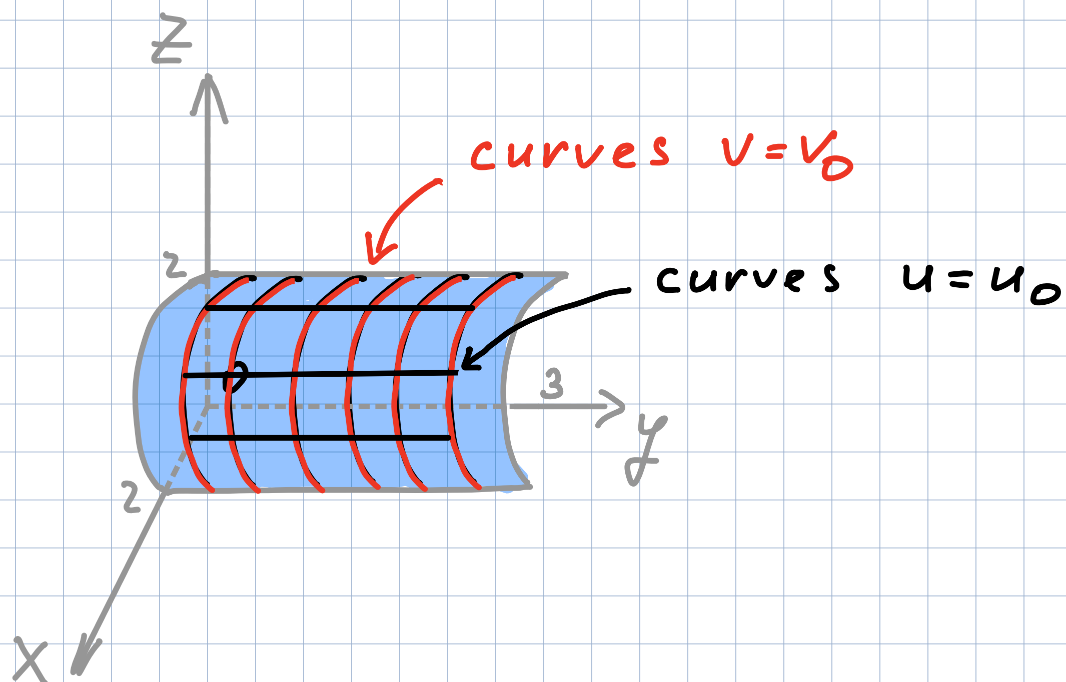
Grid curves:



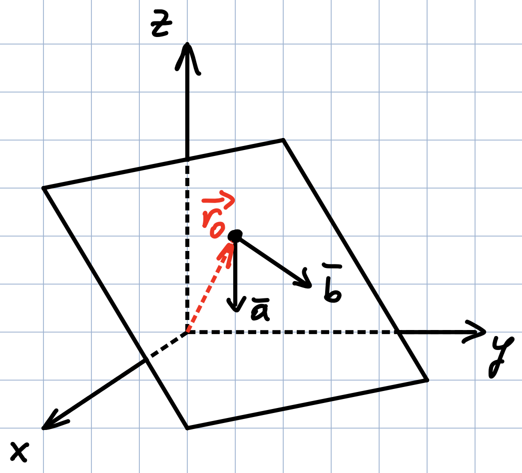
$C_1: \vec{r}(u_0, v)$

$C_2: \vec{r}(u, v_0)$

In the previous example:



Ex: Find a vector function representing the plane through P_0 containing non-parallel vectors $\vec{a}$ and $\vec{b}$.
(position vector $\vec{r_0}$.)

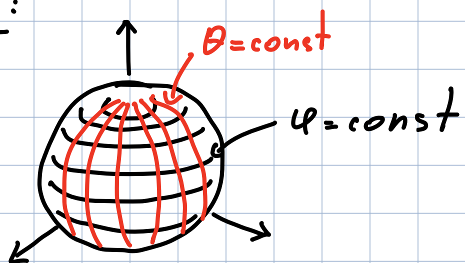


Sol: $\vec{r} = \underbrace{\vec{r_0}}_{\langle x_0, y_0, z_0 \rangle} + u\underbrace{\vec{a}}_{\langle a_1, a_2, a_3 \rangle} + v\underbrace{\vec{b}}_{\langle b_1, b_2, b_3 \rangle}$

or
$$\begin{cases} x = x_0 + ua_1 + vb_1 \\ y = y_0 + ua_2 + vb_2 \\ z = z_0 + ua_3 + vb_3 \end{cases}$$

Ex: Find a parametric representation of a sphere $x^2+y^2+z^2=a^2$.

Sol:

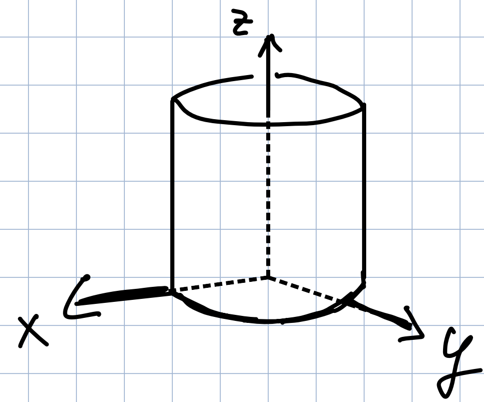


In spherical coord.: $\rho = a$

$$\left.\begin{array}{l} 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{array}\right\} \text{parameters}$$

So, we get
$$\begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases}$$
or $\vec{r}(\varphi, \theta) = \langle a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi \rangle$

Ex: Find a parametric representation of the cylinder $\begin{cases} x^2+y^2=4 \\ 0 \le z \le 1 \end{cases}$



Sol: In cylindrical coordinates, cylinder:

$r=2$ choose θ, z as parameters

$x = 2\cos\theta$

$y = 2\sin\theta$, where $0 \le \theta \le 2\pi$, $0 \le z \le 1$

$z = z$

$\vec{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z \rangle.$

Ex: Parametrize the surface $z = x^2 + 2y^2$

Sol: Choose $x = u$, $y = v$ then $\vec{r}(u,v) = \langle u, v, u^2 + 2v^2 \rangle.$

Rmk: Can do it for any graph of a function $z = f(x,y)$

$\rightsquigarrow \vec{r}(u,v) = \langle u, v, f(u,v) \rangle.$

Ex: Find a parametric representation of the cone

$$z = 2\sqrt{x^2+y^2} \quad \text{or} \quad z^2 = 4x^2 + 4y^2,\ z \geq 0$$

Sol:

$$\begin{cases} x = u \\ y = v \\ z = 2\sqrt{u^2+v^2} \end{cases}$$

$$\vec{r}(u,v) = \langle u, v, 2\sqrt{u^2+v^2} \rangle$$

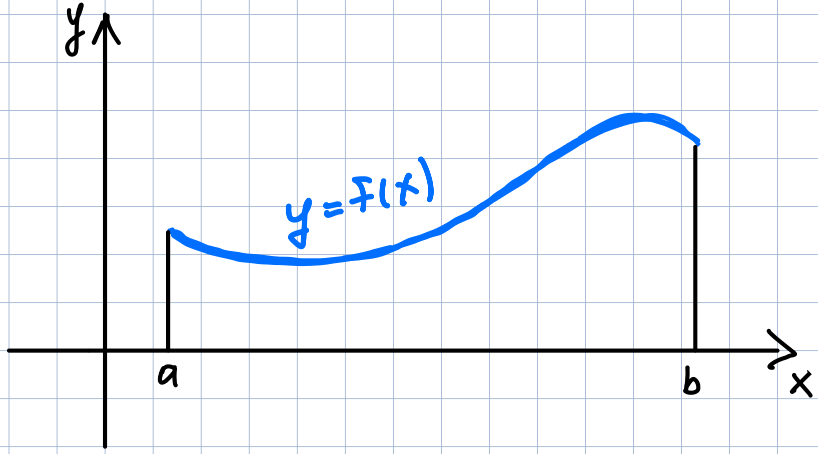
We can also choose r, θ as parameters

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= 2\sqrt{x^2+y^2} = 2r \end{aligned}$$

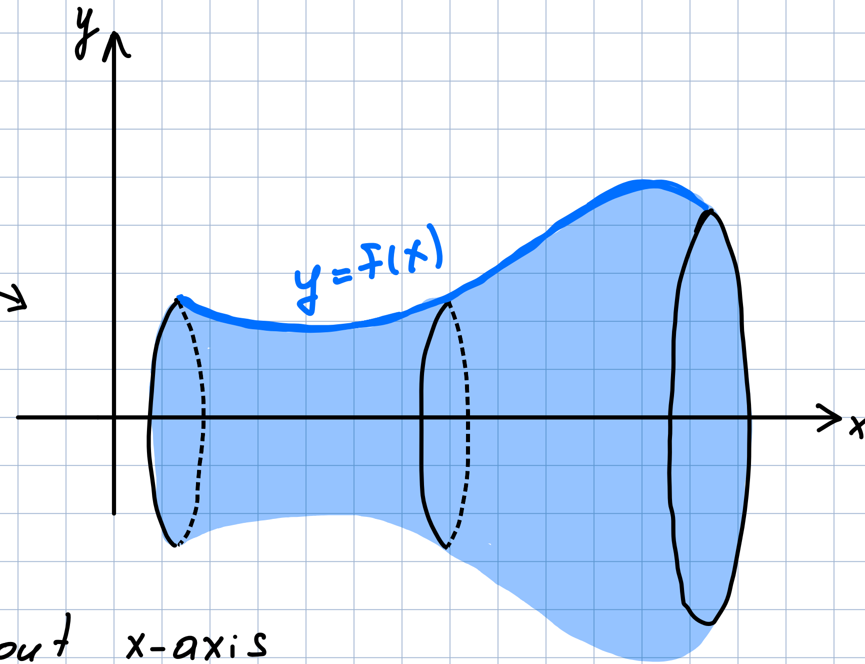
$\Rightarrow \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2r \rangle$

with $r \geq 0$, $0 \leq \theta \leq 2\pi$.

Surfaces of revolution:



rotate about x-axis



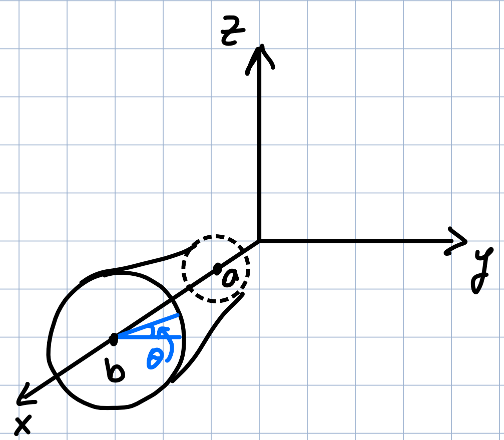
Let θ be the angle of rotation about x-axis

then $\quad x = x$

$$y = f(x)\cos\theta$$

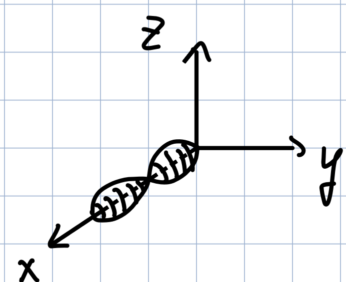
$$z = f(x)\sin\theta$$

$\left.\begin{array}{l} a \le x \le b \\ 0 \le \theta \le 2\pi \end{array}\right\}$ parameters



Ex: $y = \sin x, \quad 0 \le x \le 2\pi$.

Surface of revolution about x-axis



$$x = x$$

$$y = \sin x \cos\theta$$

$$z = \sin x \sin\theta$$